1 (a) *Determine the infimum and supremum of the set* $\{\sin(n\pi/3) : n \in \mathbf{Z}\}$ *and find a sequence from this set that converge to the infimum.*

Solution: Let $A = \{\sin(n\pi/3) : n \in \mathbf{Z}\}$. Since sine function is 2π periodic,

$$A = \{\sin(n\pi/3) : n = 0, 1, 2, 3, 4, 5\}.$$

Therefore, $\inf A = \sin(5\pi/3) = -\frac{\sqrt{3}}{2}$ and $\sup A = \sin(\pi/3) = \frac{\sqrt{3}}{2}$.
Note that $\lim\limits_{n\to\infty} \sin\frac{(6n+5)\pi}{3} = -\frac{\sqrt{3}}{2}$.

1 (b) *If* S, T *are bounded subsets of real numbers, prove that the supremum of the set* $\{s+t : s \in S, t \in T\}$ *equals* $sup(S) + sup(T)$.

Solution: See the question 1 in Analysis I (Midterm) solutions of 2012-2013.

$\square$

2 (a) *Let* $a, b > 0$. *Find the limit of the sequence* $(a^n + b^n)^{1/n}$ *as* $n \to \infty$.

Solution: Set $c = \max\{a, b\}$. Then,

$$c^n \leq a^n + b^n \leq 2\,c^n \quad \text{and thus } c \leq (a^n + b^n)^{1/n} \leq c\,(2)^{1/n}.$$

Since $2^{1/n} \to 1$ as $n \to \infty$, $(a^n + b^n)^{1/n} \to c = \max\{a, b\}$ as $n \to \infty$.

2 (b) *Prove that the sequence* $\{\sin(n)\}_n$ *has an (infinite) subsequence* $\{\sin(n_k)\}_k$ *which is completely contained in* $[1/2, 1]$.

Solution: $\sin x \in [1/2, 1]$ if and only if $x \in \bigcup\limits_{k\in\mathbf{Z}} I_k$, where $I_k = [\frac{\pi}{6} + 2k\pi, \pi - \frac{\pi}{6} + 2k\pi]$. Since lenth of $I_k = \frac{2\pi}{3} > 1$, for each $k \in \mathbf{N}$, pick $n_k \in I_k \cap \mathbf{N}$. Thus, $\sin(n_k) \in [1/2, 1]$ for all k. $\square$

3 (i) *Consider the sequence defined recursively by* $x_1 = 1, x_{n+1} = \frac{x_n^3+1}{4}$ *for all* $n \geq 1$. *Prove that* $\{x_n\}$ *is a Cauchy sequence. State what the above process yields in terms of roots of the polynomial* $x^3 - 4x + 1$.

Solution: Because $0 < x_1 \leq 1$, it follows that $0 < x_n \leq 1$ for all $n \in \mathbf{N}$. Therefore, we have

$$|x_{n+2} - x_{n+1}| = \frac{1}{4}|x_{n+1}^3 - x_n^3| = \frac{1}{4}|x_{n+1}^2 + x_{n+1}x_n + x_n^2|\,|x_{n+1} - x_n| \leq \frac{3}{4}|x_{n+1} - x_n|.$$

Therefore, (x_n) is a contractive sequence and hence (by Theorem 3.5.8, Introduction to Real Analysis by Robert G. Bartle and Donald R. Sherbert) there exists x such that $\lim x_n = x$. If we pass to the limit on both sides of the equality $x_{n+1} = \frac{x_n^3+1}{4}$, we obtain $x = \frac{x^3+1}{4}$ and hence $x^3 - 4x + 1$. Thus, x is a root of the polynomial $x^3 - 4x + 1$.

3 (ii) *For a positive integer n, consider the arithmetic and geometric means of the $n+1$ numbers $1+1/n$ (repeated n times) and 1 to deduce that the sequence $a_n = (1+1/n)^n$ is monotonically increasing. Similarly, for $n > 1$, looking at the arithmetic and geometric means of the $n+1$ numbers $1-1/n$ (repeated n times) and 1, deduce that $b_n = (1-1/n)^{-n}$ is monotonically decreasing. Finally, find a relation between b_{n+1} and a_n to deduce that both sequences $\{a_n\}$, $\{b_n\}$ converge and, converge to the same limit.*

Solution: Consider the $n+1$ numbers consists of $1+1/n$ (repeated n times) and 1. Then, their arithmetic mean (AM)= $\frac{n(1+1/n)+1}{n+1} = \frac{n+2}{n+1} = 1 + \frac{1}{n+1}$, geometric mean (GM)= $(1+1/n)^{\frac{n}{n+1}}$ and thus,

$$(1+1/n)^{\frac{n}{n+1}} \leq 1 + \frac{1}{n+1} \quad \text{i.e.,} \quad \left(1+\frac{1}{n}\right)^n \leq \left(1+\frac{1}{n+1}\right)^{n+1}.$$

Hence $a_n = (1+1/n)^n$ is monotonically increasing. Similarly, we can verify that $b_n = (1-1/n)^{-n}$ is monotonically decreasing. Since a_n is bounded above by 3 (refer Theorem 3.31 and Definition 3.30, Priniciples of Mathematical Analysis by Walter Rudin) and monotonically increasing, $\{a_n\}$ converges. Set $a = \lim_{n\to\infty} a_n$. Clearly, $\{b_n\}$ is bounded below by 0 and monotonically decreasing, $\{b_n\}$ converges to b (say). Note that

$$b_{n+1} = \left(1 - \frac{1}{n+1}\right)^{-(n+1)} = \left(\frac{n}{n+1}\right)^{-(n+1)} = \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right) a_n.$$

By taking limit on both sides of the equation $b_{n+1} = (1+\frac{1}{n})a_n$, we get that $a = b$.

3 (iii) *If $\{a_n\}$ is a sequence of positive, real numbers such that the sequence a_{n+1}/a_n converges, prove that the sequence $a_n^{1/n}$ also converges, and converges to the same limit. Give an example to show that the converse may not be true.*

Solution: Theorem 3.37 of Priniciples of Mathematical Analysis by Walter Rudin tells that if $\{a_n\}$ is a sequence of positive, real numbers such that the sequence a_{n+1}/a_n converges, then $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} a_n^{1/n}$. Example 3.35(b) of Priniciples of Mathematical Analysis by Walter Rudin provide an example in which $a_n^{1/n}$ converges but a_{n+1}/a_n does not converges. $\square$

4 (i) *Prove that the series $\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$ diverges and that $\sum_{n=2}^{\infty} \frac{1}{n\log(n)^{1.1}}$ converges.*

Solution: See Theorem 3.29, Priniciples of Mathematical Analysis by Walter Rudin.

4 (ii) *Consider the series $\sum_{n\geq 1} a_n$ where $a_n = 2^{(-1)^n n}$.*
Determine $\liminf |a_n|^{1/n}, \limsup |a_n|^{1/n}, \liminf |a_{n+1}/a_n|, \limsup |a_{n+1}/a_n|$. *What does root test give? What does ratio test give?*

Solution: Note that $a_n = 2^{-n}$ or $a_n = 2^n$ if n is odd or even respectively. Therefore, $|a_n|^{1/n} = 1/2$ or 2 and hence, $\limsup |a_n|^{1/n} = 2$, $\liminf |a_n|^{1/n} = 1/2$. It is easy to see that $|a_{n+1}/a_n| = 2^{2n+1}$ or $2^{-(2n+1)}$ depends on n is odd or even. Thus, $\liminf |a_{n+1}/a_n| = 0, \limsup |a_{n+1}/a_n| = \infty$. By Root test, we can conclude that $\sum a_n$ converges. But, Ratio test gives no conclusion.

4 (iii) *Using root/ratio/Raabe tests or otherwise, determine the convergence or otherwise of each of the following series:*

(a) $\sum_{n\geq 1} \frac{1}{\binom{2n}{n}}$;

Solution: Take $b_n = \frac{1}{\binom{2n}{n}} = \frac{n!n!}{(2n)!}$. Then,

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)(n+1)}{(2n+1)(2n+2)} \to \frac{1}{4} \text{ as } n \to \infty.$$

Thus, by Ratio test, $\sum b_n$ converges.

(b) $\sum_{n\geq 1} a_n^3$, where $a_n = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$.

Solution: Set $b_n = a_n^3$, where $a_n = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$. Note that, $\frac{a_{n+1}}{a_n} = \frac{(2n+2)}{(2n+1)}$. Now, consider

$$\begin{aligned}
\left(\left|\frac{b_n}{b_{n+1}}\right| - 1\right) n &= \left[\left(\frac{2n+2}{2n+1}\right)^3 - 1\right] n \\
&= \frac{\left[(2n+2)^3 - (2n+1)^3\right] n}{(2n+1)^3} \\
&= \frac{\left[(2n+2)^2 + (2n+1)^2 + (2n+2)(2n+1)\right] n}{(2n+1)^3}.
\end{aligned}$$

Thus, $\left(\left|\frac{b_n}{b_{n+1}}\right| - 1\right) n \to \frac{3}{2}$ as $n \to \infty$. Thus, by Rabbe's test, $\sum b_n$ converges.

□

5.1 (a) *Define the interior S^0 and closure $\bar{S}$ of a set $S \subset \mathbf{R}$.*

Solution: See the Definition 2.18 of Priniciples of Mathematical Analysis by Walter Rudin

5.1 (b) *If $S_n (n \geq 1)$ are subsets of $\mathbf{R}$, then prove that $(\bigcap_{n=1}^{\infty} S_n)^0 \subseteq \bigcap_{n=1}^{\infty} S_n^0$. Give an example to show that the inclusion could be proper.*

Solution: Let $x \in (\bigcap_{n=1}^{\infty} S_n)^0$. Then there exist a neighbourhood U of x such that $U \subseteq \bigcap_{n=1}^{\infty} S_n$. Inparticular, $U \subseteq S_n$ for all n and therefore $x \in S_n^0$ for all n. Hence $(\bigcap_{n=1}^{\infty} S_n)^0 \subseteq \bigcap_{n=1}^{\infty} S_n^0$.

To show this inclusion may be proper, consider the following example. Take $S_n = (-1/n, 1/n)$. Then $S_n^0 = S_n$ for all n. Observe that,

$$\bigcap_{n=1}^{\infty} S_n^0 = \{0\} \text{ and } (\bigcap_{n=1}^{\infty} S_n)^0 = (\{0\})^0 = \emptyset.$$

Thus $(\bigcap_{n=1}^{\infty} S_n)^0 \subsetneq \bigcap_{n=1}^{\infty} S_n^0$.

5.1 (c) *For any subset S of $\mathbf{R}$, prove that $S^0 = \overline{(S^c)}^c$ where A^c denotes the complement of A.*

Solution: Note that A^0 is the largest open set contained in A and $\bar{A}$ is the smallest closed set that contains A.

Since $S^0 \subseteq S$, we have $S^c \subseteq (S^0)^c$ = closed set (since S^0 is open). Thus, $\overline{S^c} \subseteq (S^0)^c$ and therefore $S^0 \subseteq \overline{(S^c)}^c$.

Other way inclusion: Since $\overline{(S^c)}$ is a closed set containing S^c, $\overline{(S^c)}^c$ is a open set contained in S. Thus, $\overline{(S^c)}^c \subseteq S^0$. Hence proved.

5.2 (a) *Write down a set of open intervals $I_n (n \geq 1)$ which cover $(0,1)$ such that finitely many of the I_n's do not cover $(0,1)$.*

Solution: Take $I_n = (1/n, 1)$ so that $(0,1) = \bigcup_{n=1}^{\infty} I_n$. Now, it is trivial to see that finitely many of the I_n's do not cover $(0,1)$.

5.2 (b) *Suppose S is a set of real numbers such that whenever $S \subset \bigcup_{n\geq 1} U_n$ with U_n's open, there is a positive integer n such that $S \subset \bigcup_{m=1}^{n} U_m$. Prove that S must be closed. i.e., Prove that compact sets are closed.*

Solution:

See Theorem 2.34, Priniciples of Mathematical Analysis by Walter Rudin.

$\square$